Principle of Least Time in Optics

Rick Dower Boston QuarkNet 11/25/2024

**REFLECTION – LEAST DISTANCE**

Claudius Ptolemy (c. 100 – c. 170 CE), in his *Optics*, noted that light rays appear to travel in straight lines and a light ray incident on a mirror and reflected from the mirror “makes equal angles with the perpendicular to the mirror at that point [of reflection].”[[1]](#footnote-1) He then describes experiment swith plane, convex, and concave mirrors[[2]](#footnote-2) that demonstrate the equality of angles. This equality is traditionally called the Law of Reflection.

Heron (or Hero) of Alexandria (*c*. 100 CE), in his *Catoptrics*, wondered why light rays travel in straight lines and “why the reflections are at equal angles [to the incident rays].”[[3]](#footnote-3) He argued and light “rays are emitted with infinite velocity. Therefore, they [like swift arrows] . . . will move along the shortest path, a straight line.”[[4]](#footnote-4) Then he showed that light from an object seen in a mirror takes the shortest path between object, mirror, and eye, as in the following proof.

O

E

Let AB represent a plane mirror, point E the eye, point O the object seen in the mirror, R the point on the mirror at which light from O reflects to the eye, Q another point on the mirror.

A

B

T

S

R

Q

Extend ER beyond the mirror. Draw OS perpendicular to the mirror and extend it to cross the extension of ER at T.

$∠ERA=∠ORS$ Complements of equal law of reflection angles

$∠ERA=∠TRS$ Vertical angles

$∠OSR=∠TSR$ Perpendicular angles

Side RS is common to DRSO and DRST.

DRSO $≅$ DRST Angle-Side-Angle

RO = RT and SO = ST Congruent triangles

Side QS is common to DQSO and DQST.

DQSO $≅$ DQST Side-Angle-Side

QO = QT Congruent triangles

But ERT is a straight line and the shortest distance between E and T.

Thus, (ER + RT) = (ER + RO) < (EQ + QO) = (EQ+QT)

Therefore, Light travels from O to R to E (or E to R to O) by the shortest distance.

We can do the proof in the reverse direction with Cartesian coordinates and calculus to show the minimum distance path for light reflection yields the Law of Reflection.

For light that reflects off a surface, suppose the medium for y>0 in the diagram below is uniform and a mirror surface lies along the x-axis. The line SR is perpendicular to the *x*-axis.

Consider light from point P (0, *y*P) that reflects from mirror point R (*x*R, 0) to point Q (*x*Q, *y*Q).

*y*

S

P

Length PR = $\sqrt{x\_{R}^{2}+y\_{P}^{2}}=\left[x\_{R}^{2}+y\_{P}^{2}\right]^{1/2}$

Length RQ = $\left[\left(x\_{Q}-x\_{R}\right)^{2}+y\_{Q}^{2}\right]^{1/2}$

*q*i

Q

*q*r

*q*2

*q*1

*x*

*L* = PR + RQ = $\left[x\_{R}^{2}+y\_{P}^{2}\right]^{1/2}+\left[\left(x\_{Q}-x\_{R}\right)^{2}+y\_{Q}^{2}\right]^{1/2}$

R

To find the value of *x*R for minimum total distance, *L*, set the derivative *dL*/*dx*R equal to zero.

$$\frac{dL}{dx\_{R}}=\frac{1}{2}\left[x\_{R}^{2}+y\_{p}^{2}\right]^{-{1}/{2}}\left(2x\_{R}\right)+\frac{1}{2}\left[\left(x\_{Q}-x\_{R}\right)^{2}+y\_{Q}^{2}\right]^{-{1}/{2}}\left(-2\right)\left(x\_{Q}-x\_{R}\right)=0$$

This simplifies to

$$\frac{x\_{R}}{\left(x\_{R}^{2}+y\_{P}^{2}\right)^{{1}/{2}}}-\frac{\left(x\_{Q}-x\_{R}\right)}{\left[\left(x\_{Q}-x\_{R}\right)^{2}+y\_{Q}^{2}\right]^{{1}/{2}}}=0$$

Equivalently, cos*q*1 – cos*q*2 = 0 or cos*q*1 = cos*q*2

Since, $0\leq θ\_{1}\leq 90^{∘} and 0\leq θ\_{2}\leq 90^{∘}$, we have *q*1 = *q*2 and, by equal complements of right angles,

Law of Reflection: *q*i = *q*r or Angle of incidence = Angle of reflection.

REFRACTION – LEAST TIME

Claudius Ptolemy was the first to record measurements of light refraction from air to water, air to glass, and water to glass. His *Optics* contains what I consider the first lab report. Ptolemy gave a description of his apparatus, three data tables, mathematical fits to his data, and a general conclusion from his experiments. Seeking a mathematical pattern in his refraction measurements, Ptolemy accounted for measurement uncertainty by regularizing his data, as was traditional with astronomical data at the time, in the following tables from *Optics*.[[5]](#footnote-5)

Additional columns list refraction angles calculated (*q*c) from the Law of Refraction and index of refraction values and the difference (D) between Ptolemy’s values and modern values.

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
|  |  |  | *n*aw =  |  |
|   **Ptolemy‘s Data: Air-Water** | 1.33 |  |
| *q*1 | *q*2 |  | *q*c | D = *q*2 - *q*c |
| (deg) | (deg) |  | (deg) | (deg) |
| 0 | 0.0 |  | 0 | 0 |
| 10 | 8.0 |  | 7.50 | 0.50 |
| 20 | 15.5 |  | 14.90 | 0.60 |
| 30 | 22.5 |  | 22.08 | 0.42 |
| 40 | 29.0 |  | 28.90 | 0.10 |
| 50 | 35.0 |  | 35.17 | -0.17 |
| 60 | 40.5 |  | 40.63 | -0.13 |
| 70 | 45.5 |  | 44.95 | 0.55 |
| 80 | 50.0 |  | 47.77 | 2.23 |

Ptolemy’s refraction angles for air - water are given by *q*2 = 0.825\**q*1 – 0.0025\**q*12 .

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
|  |  |  | *n*ag =  |  |
|   **Ptolemy’s Data: Air-Glass** | 1.5 |  |
| *q*1 | *q*2 |  | *q*c | D = *q*2 - *q*c |
| (deg) | (deg) |  | (deg) | (deg) |
| 0 | 0.0 |  | 0 | 0 |
| 10 | 7.0 |  | 6.65 | 0.35 |
| 20 | 13.5 |  | 13.18 | 0.32 |
| 30 | 19.5 |  | 19.47 | 0.03 |
| 40 | 25.0 |  | 25.37 | -0.37 |
| 50 | 30.0 |  | 30.71 | -0.71 |
| 60 | 34.5 |  | 35.26 | -0.76 |
| 70 | 38.5 |  | 38.79 | -0.29 |
| 80 | 42.0 |  | 41.04 | 0.96 |

Ptolemy’s refraction angles for air - glass are given by *q*2 = 0.725\**q*1 – 0.0025\**q*12 .

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
|  |  |  | *n*wg =  |  |
|   **Ptolemy’s Data: Water-Glass** | 1.125 |  |
| *q*1 | *q*2 |  | *q*c | D = *q*2 - *q*c |
| (deg) | (deg) |  | (deg) | (deg) |
| 0 | 0.0 |  | 0 | 0 |
| 10 | 9.5 |  | 8.88 | 0.62 |
| 20 | 18.5 |  | 17.70 | 0.80 |
| 30 | 27.0 |  | 26.39 | 0.61 |
| 40 | 35.0 |  | 34.85 | 0.15 |
| 50 | 42.5 |  | 42.92 | -0.42 |
| 60 | 49.0 |  | 50.34 | -1.34 |
| 70 | 56.0 |  | 56.65 | -0.65 |
| 80 | 62.0 |  | 61.09 | 0.91 |

Ptolemy’s refraction angles for water - glass are given by *q*2 = 0.975\**q*1 – 0.0025\**q*12 .

Ptolemy’s measurements are nearly all within 1 degree of values computed from the index of refraction of the materials indicated in the tables. Though Ptolemy gave no physical rationale for his numerical patterns, he did note qualitatively that greater refraction occurs for substances with greater differences in density.[[6]](#footnote-6)

The earliest equivalent version of our modern Law of Refraction occurs in Ibn Sahl’s *On Burning Instruments* (*c*. 980 CE).[[7]](#footnote-7) However, this insight, from Islam’s Golden Age, had no subsequent influence. English explorer and mathematician Thomas Herriot (c. 1560 – 1621) and Dutch mathematician Willebrord Snell (1580 – 1626) are credited with unpublished statements of Law of Refraction equivalents in 1601 and 1621, respectively.[[8]](#footnote-8) However, the first published statement and rationale for the Law of Refraction is found in René Descartes’ *La Dioptrique* (1637). There Descartes uses an image of a tennis ball struck toward a thin cloth, represented by the ground (BE) in the figure,[[9]](#footnote-9) that reduces the ball’s speed in the vertical direction but not in the horizontal direction to explain refraction when light travels from one substance to another. In this case, he notes that the ball would bend away (BI) from the normal (HG) to the interface between the substances. Thus, he argues that light must receive a boost in speed when traveling from air to water or another dense substance when it bends toward the normal. Descartes’ reasoning is somewhat confused since elsewhere he argues that light travels with infinite speed.

Isaac Newton (1642 – 1726) accepted Rømer’s observations of the eclipses of Jupiter’s moon Io as demonstrating that light has a finite velocity. In *Principia* Proposition 94 and 95,[[10]](#footnote-10) he demonstrated that a particle that experienced only a force perpendicular to a surface would move with the ratio of the sines of angles to the normal inverse to the ratio of velocities before and after crossing the surface. In his *Opticks*, Newton assumes the standard Law of Refraction in Axiom V: ”The Sine of Incidence is either accurately [for single colors] or very nearly [for a mix of colors, as in white light] in a given [constant] Ratio to the Sine of Refraction.”[[11]](#footnote-11) He subsequently argued “*That Bodies refract light by acting upon its rays by acting in lines perpendicular to their surfaces*.”[[12]](#footnote-12) Newton’s prestige was such that scientists generally accepted his particle model of light during the 18th century even though it required one to believe that light travels faster in a denser medium than in a less dense medium. A version of Newton’s demonstration follows.

 Assume a light ray consists of a stream of particles that experience a force perpendicular to the interface surface when light travels from one medium (1) to another medium (2).

 **v**1 Medium **2**

 **v**1 ⎢⎢ θ1 **v**2 ⊥ > **v**1 ⊥

 **v**1 ⊥ θ2 **v**2 ⎢⎢ = **v**1 ⎢⎢

 Medium **1** **v**2

 Split the velocity vectors into components parallel (**v** ⎢⎢) and perpendicular (**v**⊥) to the surface separating the two media in which light travels. Assume a force acts on the light particles in the direction perpendicular to the surface so that the **v**⊥ value changes. However, no force acts parallel to the surface so that **v**1 ⎢⎢ = **v**2 ⎢⎢.

From the diagram above one can see that
 sinθ1 = **v**1 ⎢⎢/**v**1 and sinθ2 = **v**2 ⎢⎢/**v**2

So **v**1 ⎢⎢ = **v**1sinθ1 = **v**2sinθ2 = **v**2 ⎢⎢

Then the relative index of refraction from medium 1 to medium 2 is

Newton’s Particle Model of Refraction: 

For example, the Newtonian prediction for light going from air to water is

.

 Newton could have explained the law of refraction by supposing that particles traveling from a lower index medium (**1**) to a higher index medium (**2**) experience a force that reduces the particle’s velocity component parallel to the surface.

 ***v****1* ⎢⎢ ***v***1 Medium **2**

 ***v*** ⊥ ***v*** ⊥

  ***v***2 ⎢⎢ < ***v***1 ⎢⎢

 Medium **1** ***v***2

 However, suppose several light rays from different directions hit the same spot on the surface at the same time. Then the force at the surface would have to be oriented in different directions for different light particles. It is hard to imagine how a simple force at the surface could be applied in the appropriate direction for light particles incident from different directions at the same time. In addition, keeping **v** ⊥ constant would yield $\frac{\cos(θ\_{1})}{\cos(θ\_{2})}=constant$, which is not observed

The great Dutch scientist Christiaan Huygens (1629-1695) also recognized Rømer’s observations. Although he used Rømer’s flawed estimate of 11 minutes, rather than the actual 8 minutes 20 seconds required for light to travel from Sun to Earth, Huygens was the first to calculate the speed of light in space.[[13]](#footnote-13) Huygens proposed a wave model to explain light behavior in his *Treatise on Light* (1690). Rather than a tennis ball or a particle, Huygens’ analogy for light was sound waves in air, which can traverse one another without hinderance, as light rays do.[[14]](#footnote-14) To explain reflection and refraction, Huygens argued that light rays correspond to a succession of plane waves with the travel direction perpendicular to the wave crests. A version of his account of light refraction[[15]](#footnote-15) follows.

 Assume a light ray consists of a series of waves traveling outward from the source with wave crests perpendicular to the wave travel direction, as in the illustration.

 Point source • Wave point source • ) ) ) ) ) ) ) )

 ↑ light ray ↑ wave crests

 A wave train in an homogeneous isotropic medium will travel with a constant velocity (*v*wave), wavelength (λ),l and frequency (*f* ) related by

 *v*wave = λ*f*

 Suppose a light ray traveling across a boundary between two different substances is analogous to a wave train traveling from a region with one wave velocity to a region with a different wave velocity, where *v*1 = wave speed in region 1 and *v*2 = wave speed in region 2.

The part of a wave crest that hits the boundary between regions first will change speeds first. This leads to a kink in the wave crest at the boundary and a change in the wave travel direction (perpendicular to the wave crest) at the boundary as illustrated below.

 Medium 1 refracted wave

 travel direction λ2

 θ1

 θ2

 *x*

 Medium 2 λ1

 incident wave

 travel direction

 ***x*** = crest-to-crest distance along boundary

 sinθ1 = λ1/*x* and sinθ2 = λ2/*x*

The frequency (*f* ), number of waves per second at a point, is the same in both mediums.

Then the relative index of refraction from medium 1 to medium 2 is



Huygens’ Wave Model of Refraction: 

For example, the wave model prediction for light going from air to water is

.

Pierre de Fermat (1607 – 1665) trained as a lawyer and served as a councilor (lawyer) at the Parlement de Toulouse from 1630 to the end of his life. He devoted much of his spare time to mathematics. In 1657 he received a letter from a friend who noted that Heron’s principle of least distance explained light reflection but not light refraction. Five years later (1662) Fermat replied in a letter in which he showed that light refraction could be explained if one assumed that light traveled slower in a higher refraction index medium, and light took the path that was the least travel time[[16]](#footnote-16), not least distance. His idea met resistance because it conflicted with Descartes’ explanation of refraction. For his proof, Fermat was able to use his investigations in finding the slopes of tangent lines. With the 19th century revival the wave theory of light, Fermat’s principle gained interest. A modern proof follows.

Suppose light travels from a source at (0, y1) in Medium 1 (y>0) to (x2, y2) in Medium2 (y<0).
We want to find the path (value of *x*) for least travel time.

*y*1

Medium 1 (*y*>0), index of refraction = *n*1, light speed = *v*1 = *c*/*n*1

*q*1

*x*2

0

*x*

Medium 2 (*y*<0), index of refraction = *n*2, light speed = *v*2 = *c*/*n*2

*y*2

q2

*T* = travel time for light from (0, *y*1) to (*x*,0) to (*x*2, *y*2).

$$T=\frac{\left[x^{2}+y\_{1}^{2}\right]^{{1}/{2}}}{v\_{1}}+\frac{\left[\left(x\_{2}-x\right)^{2}+y\_{2}^{2}\right]^{{1}/{2}}}{v\_{2}}$$

To find the *x*-value for the minimum time, set the derivative *dT*/*dx* = 0.

$$\frac{dT}{dx}=\frac{\frac{1}{2}\left[x^{2}+y\_{1}^{2}\right]^{{-1}/{2}}\left(2x\right)}{v\_{1}}+\frac{\frac{1}{2}\left[\left(x\_{2}-x\right)^{2}+y\_{2}^{2}\right]^{{-1}/{2}}\left(2\right)\left(x\_{2}-x\right)\left(-1\right)}{v\_{2}}=0$$

Rearranging, we get

$$\frac{1}{v\_{1}}\left[\frac{x}{\left(x^{2}+y\_{1}^{2}\right)^{{1}/{2}}}\right]=\frac{1}{v\_{2}}\left[\frac{\left(x\_{2}-x\right)}{\left[\left(x\_{2}-x\right)^{2}+y\_{2}^{2}\right]^{{1}/{2}}}\right]$$

$$\frac{\sin(θ\_{1})}{v\_{1}}=\frac{\sin(θ\_{2})}{v\_{2}}$$

$$\frac{\sin(θ\_{1})}{\sin(θ\_{2})}=\frac{v\_{1}}{v\_{2}}=\frac{{c}/{n\_{1}}}{{c}/{n\_{2}}}=\frac{n\_{2}}{n\_{1}}$$

This is the expected Law of Refraction for one color of light. Measurements of light speed in the mid-19th century by Fizeau and Foucault for several substances confirmed the speed measurements expected from the wave model of refraction and, thus, the least-time principle of Fermat.

Note as a corollary that for a uniform medium, *n*1 = *n*2 implies that *q*1 = *q*2 and light travels in a straight line, as expected. Thus, the principle of least time, a consequence of wave motion, can explain straight-line travel, mirror reflection, and refraction for light.

1. M. Cohen and I. Drabkin, *A Source Book in Greek Science*, Harvard University Press, Cambridge, Massachusetts, 1948, p. 269. [↑](#footnote-ref-1)
2. Ibid. pp. 270-271. [↑](#footnote-ref-2)
3. Ibid*.*, p. 263. [↑](#footnote-ref-3)
4. Ibid. [↑](#footnote-ref-4)
5. Ibid., pp. 275-277. [↑](#footnote-ref-5)
6. Ibid., p. 279. [↑](#footnote-ref-6)
7. A. Mark Smith, *From Sight to Light*, The University of Chicago Press, Chicago, 2015, pp. 176-178. [↑](#footnote-ref-7)
8. Ibid., p. 293 n. 40. [↑](#footnote-ref-8)
9. William F. Magie, *A Source Book in Physics*, Harvard University Press, Cambridge, Massachusetts, 1965, p. 268. [↑](#footnote-ref-9)
10. Isaac Newton, The *Principia*, 3rd ed. (1726), I. Bernard Cohen and Anne Whitman, trans., University of California Press, Berkeley, 1999, pp. 622-624. [↑](#footnote-ref-10)
11. Isaac Newton, *Opticks*, 4th ed. (1730), Dover Publications, Inc., 1952, p. 5. [↑](#footnote-ref-11)
12. Ibid., pp. 79-80. [↑](#footnote-ref-12)
13. Christiaan Huygens, *Treatise on Light* (1690), S. Thompson, trans., Dover Publications, Inc., New York, 1962., pp. 7-10. [↑](#footnote-ref-13)
14. Ibid., pp.3-4. [↑](#footnote-ref-14)
15. Ibid., pp. 34-38. [↑](#footnote-ref-15)
16. Magie, pp. 278 – 280. [↑](#footnote-ref-16)