Least Time Principle and Geometric Optics

Boston QuarkNet Meeting

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11/25/2024

Law of Reflection

- Euclid (*c*. 300 BCE), in his *Optics,* was the first to combine the idea that light traveled in straight lines (emitted from the eyes) and geometry to describe visual perception.
- Claudius Ptolemy (*c*. 100 – *c*. 170 CE), in his *Optics*, noted that a light ray reflected from a mirror "makes equal angles with the perpendicular to the mirror at the point [of reflection]. Ptolemy then described experiments with plane, convex, and concave mirrors that demonstrated the equality of angles.
- <u>Law of Reflection</u>: Angle of Incidence = Angle of Reflection

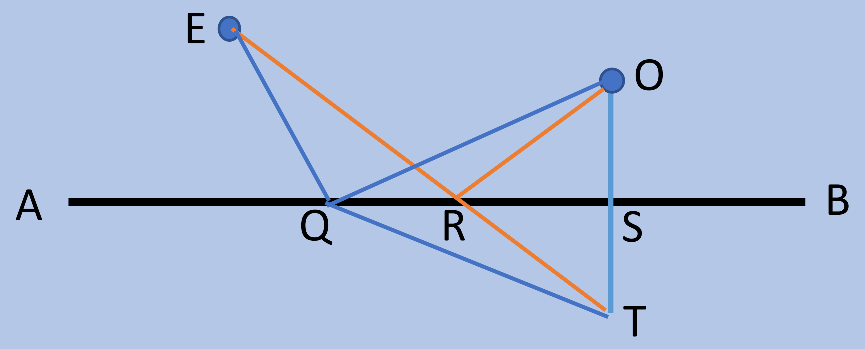
Why Does Light Reflect at an Equal Angle?

- Heron (or Hero) of Alexandria (*c*. 100 CE), in his *Catoptrics*, noted that faster objects travel along straighter lines. Light, he argued, has an infinite speed. So, light takes a straight-line path, the shortest distance from point to point in a uniform medium.

- For reflection then, light also follows the shortest path. He showed geometrically that the Law of Reflection yields the shortest path for light between an object point, a mirror, and the eye.

Proof

Line AB represents a plane mirror,
O the object point, E the eye,
R the reflection point,
Q another mirror point,
T the extension of ER,
OS perpendicular to the mirror.
i, *r* are incidence, reflection angles



$\angle \mathrm{ERA} = \angle \mathrm{ORS}$ Complements to $\angle i, \angle r$
$\angle \mathrm{ERA} = \angle \mathrm{TRS}$ Vertical angles

$\angle \mathrm{OSR} = \angle \mathrm{TSR}$ Perpendicular angles

Side RS common to ΔRSO and ΔRST

$\Delta \mathrm{RSO} \cong \Delta \mathrm{RST}$ Angle-Side-Angle

RO = RT, SO = ST Congruent triangles

Side QS common to ΔQSO and ΔQST

QO = QT Congruent triangles

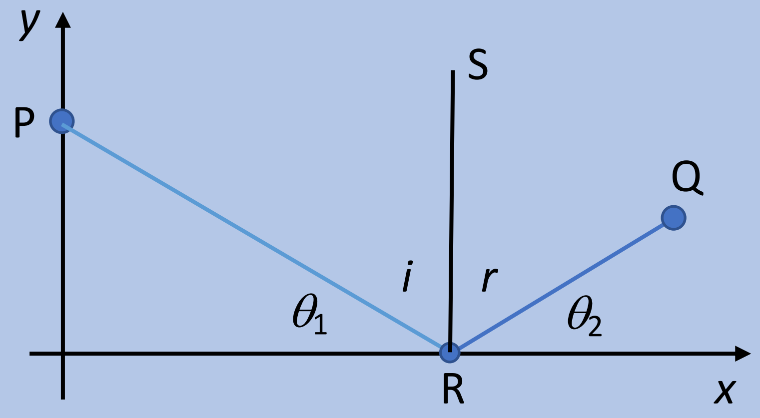
ERT is a straight line, shortest distance

(ER+RT) = (ER+RO) < (EQ+QO) = (EQ+QT)

<u>ERO shortest distance from E to AB to O</u>

Reverse Direction Proof

Light from P $(0, y_P)$ reflects from mirror at R $(x_R, 0)$ to Q (x_Q, y_Q).



SR perpendicular to x-axis.

PR = $[x_R^2 + y_P^2]^{1/2}$

RQ = $\left[\left(x_Q - x_R\right)^2 + y_Q^2\right]^{1/2}$

Let $L(x_R)$ = PR + RQ.

For minimum L, $\frac{dL}{dx_R} = 0.$

$$\frac{1}{2}[x_R^2 + y_P^2]^{-1/2}(2x_R) + \frac{1}{2}\left[\left(x_Q - x_R\right)^2 + y_Q^2\right]^{-1/2}(2)\left(x_Q - x_R\right)(-1) = 0$$

Simplifying, we get

$$\frac{x_R}{[x_R^2+y_P^2]^{1/2}} = \frac{\left(x_Q-x_R\right)}{\left[\left(x_Q-x_R\right)^2+y_Q^2\right]^{1/2}},$$

or $\cos\theta_1 = \cos\theta_2$, and $\theta_1 = \theta_2$.

$\theta_1 + i = 90° = r + \theta_2$. <u>Thus, $i = r$</u>.

Ptolemy – the First Lab Report (1)

In addition to confirming the Law of Reflection, Ptolemy was the first to record measurements of refraction from air to water, air to glass, and water to class.

He gave a
- description of his apparatus,
- three data tables,
- mathematical fits to the data,

and his general conclusion.

Ptolemy's Data for Air-Water

Θ_1 (air)	Θ_2(water)
(deg)	(deg)
0	0.0
10	8.0
20	15.5
30	22.5
40	29.0
50	35.0
60	40.5
70	45.5
80	50.0

Ptolemy – the First Lab Report (2)

Ptolemy regularized his data, the common practice with astronomical data at the time, and found a pattern:

$\theta_2 = 0.825*\theta_1 - 0.0025*\theta_1^2$.

The the table θ_c is calculated from the standard law of refraction $\frac{\sin\theta_a}{\sin\theta_w} = n_{aw}$, where θ_a and θ_w are the angles to the perpendicular in air and water, respectively.

			n_{aw} =	
Ptolemy's Data: Air-Water			1.33	
θ_a	θ_w		θ_c	$\Delta = \theta_w - \theta_c$
(deg)	(deg)		(deg)	(deg)
0	0.0		0	0
10	8.0		7.50	0.50
20	15.5		14.90	0.60
30	22.5		22.08	0.42
40	29.0		28.90	0.10
50	35.0		35.17	-0.17
60	40.5		40.63	-0.13
70	45.5		44.95	0.55
80	50.0		47.77	2.23

Ptolemy – the First Lab Report (3)

For his measurements of air-to-glass and water-to-glass refraction, Ptolemy found similar patterns.

He concluded, qualitatively, that greater refraction occurs for substances with greater differences in density.

Optics in Islam’s Golden Age

During Islam’s Golden Age fo science (*c*. 750 – 1250 CE), scholars preserved and commented upon the ancient Greek texts.

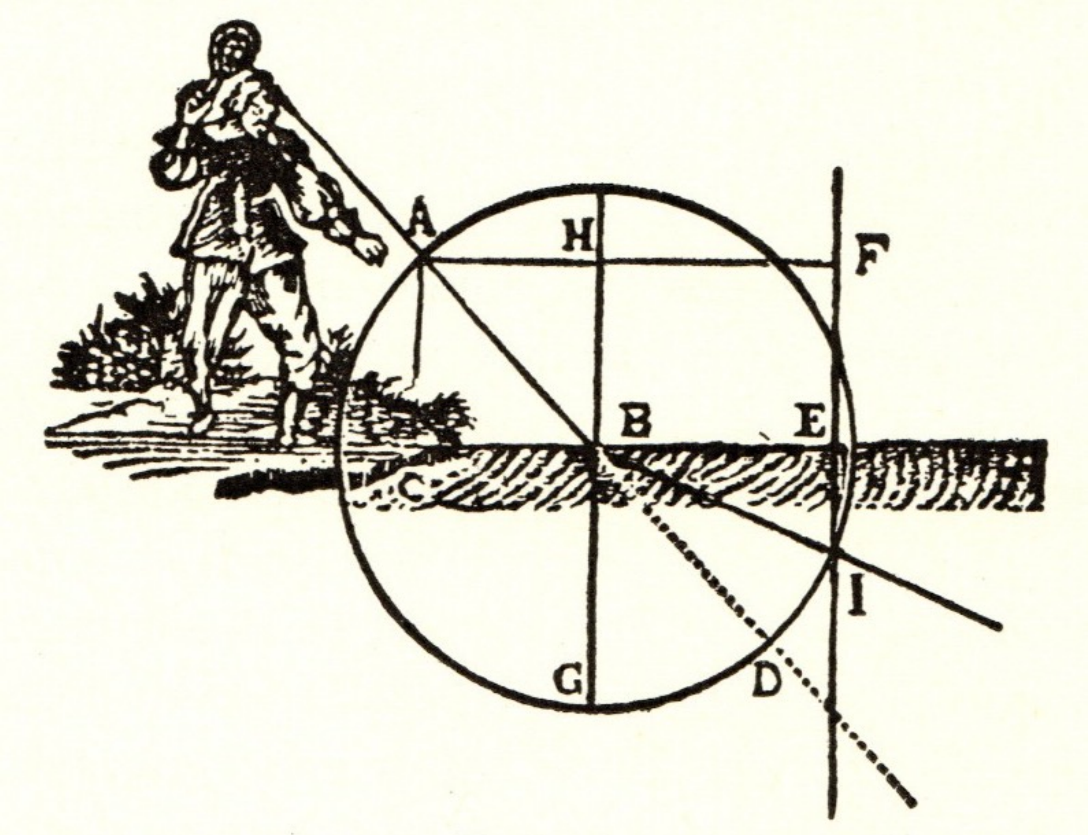
Ibn Sahl’s *On Burning Instruments* (c. 980 CE) contains an equivalent version of our modern law of refraction, but it had no subsequent influence.

Ibn al-Haytham (Alhazan), in his *Book of Optics* (c. 1020 CE), showed
(1) vision occurs by light reflecting from an object to the eye,
(2) incident ray, reflected ray, and mirror normal are in same plane,
(3) gave the first clear description and analysis of the *camera obscura,*
(4) followed Ptolemy in measuring refraction, among many advances.

Descartes and Law of Refraction (1)

Thomas Herriot (1601) and Willebrord Snell (1621) are credited with unpublished statements of the modern law of refraction.

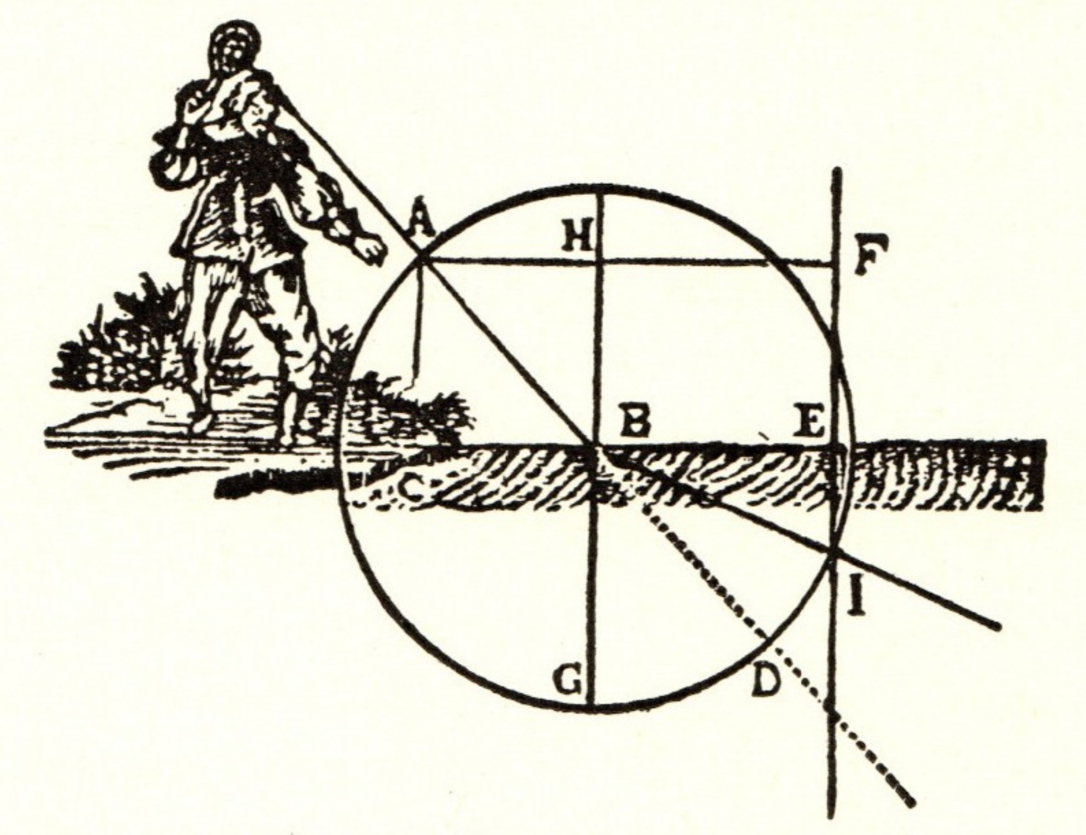
The first published statement and rationale for the Law of Refraction is contained in René Descartes' *La Dioptrique* (1637). Descartes imagined a tennis ball hit toward (AB) a thin cloth barrier (BE) in the figure.



Descartes and Law of Refraction (2)

If the cloth slowed the perpendicular component of the tennis ball's motion but did not change the horizontal component, the Law of refraction would result with the ball bending away (BI) from the surface normal.

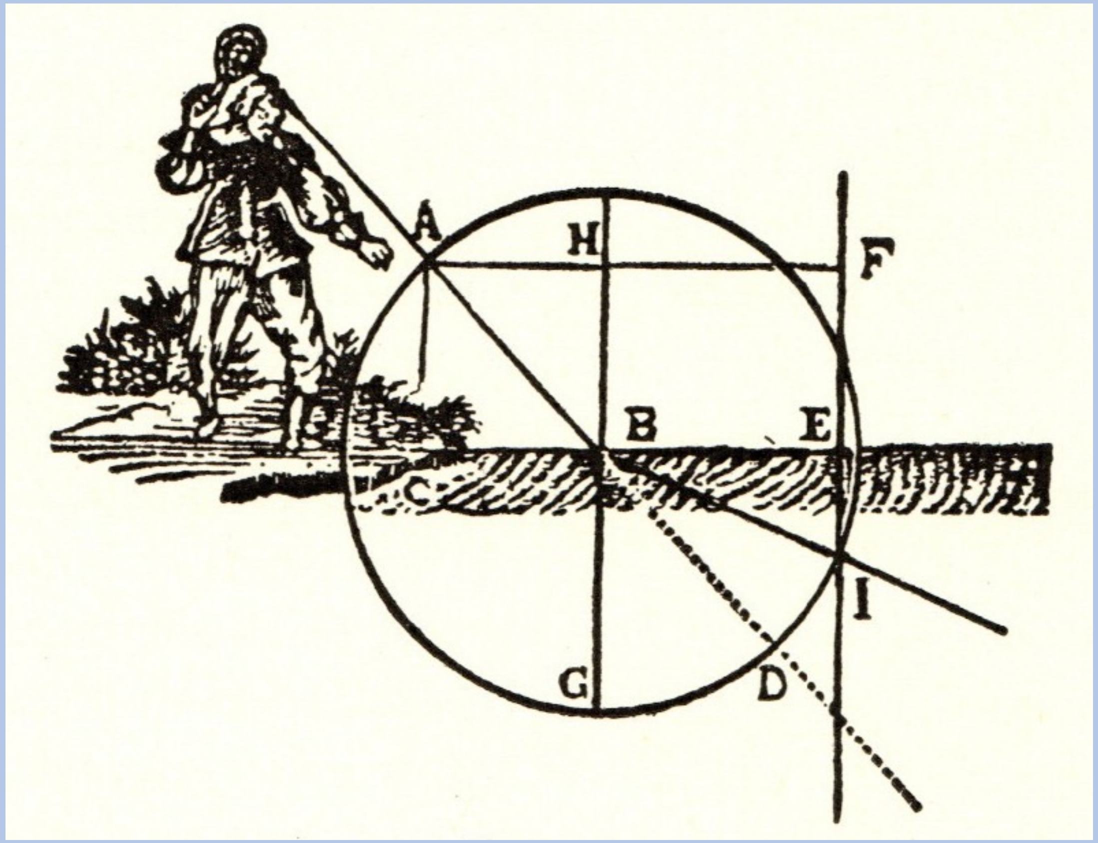
To imitate light traveling from air to water, the ball would have to be given a boost in perpendicular speed at the boundary, as Descartes noted.



Descartes and Law of Refraction (3)

Descartes argued that the law of refraction is that the lengths AH and HF are in constant ratio for light traveling from one substance to another. This ratio is equivalent to the ratio of sines of angles to the perpendicular to the surface.

Elsewhere Descartes argues that light travels with infinite velocity, which confuses his argument.



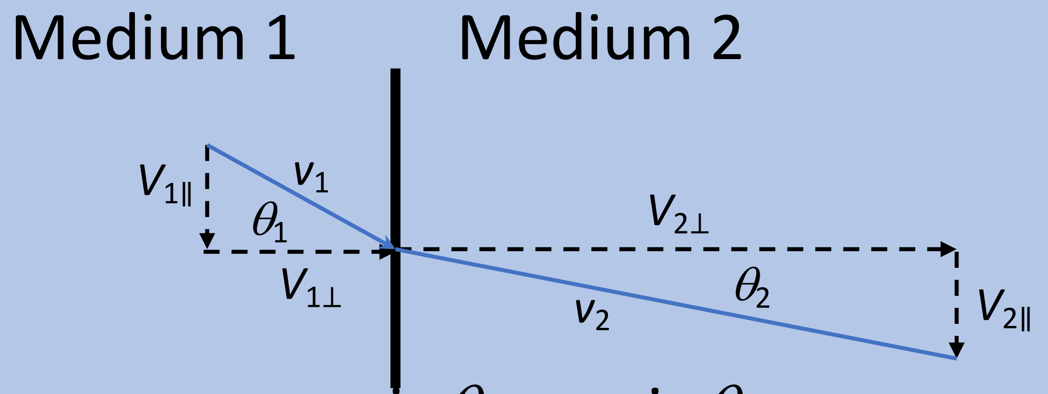
Newton's Particle Model of Refraction

In his *Principia* (1687) Newton acknowledged Rømer's observation (1676) of finite light speed.

In Propositions 94 and 95, Newton demonstrated that a particle experiencing a force perpendicular, and not parallel, to the surface at a surface boundary would change speed in Medium 2 so that

$$\frac{\sin\theta_1}{\sin\theta_2} = \frac{v_2}{v_1} = n_{12} = \text{constant},$$

which is the Law of Refraction for one color of light.



$$v_{1\parallel} = v_1 \sin\theta_1 = v_2 \sin\theta_2 = v_{2\parallel}$$

$$\frac{\sin\theta_1}{\sin\theta_2} = \frac{v_2}{v_1} = n_{12} = \text{constant},$$

independent of θ_1 for index of refraction n_{12} from medium 1 to medium 2.

Why not a force parallel to the boundary?

Could Newton have explained refraction by supposing that light particles experienced a force parallel to the surface boundary?

Suppose several light rays from different directions hit the same place on the boundary at the same time. How could the surface exert forces in different directions on different light particles at the same time?

Also, the perpendicular component of particle speed would remain constant, rather than the parallel component.

Then the expected relation relation would be $\frac{\cos\theta_1}{\cos\theta_2} = \frac{v_2}{v_1} = \text{constant},$

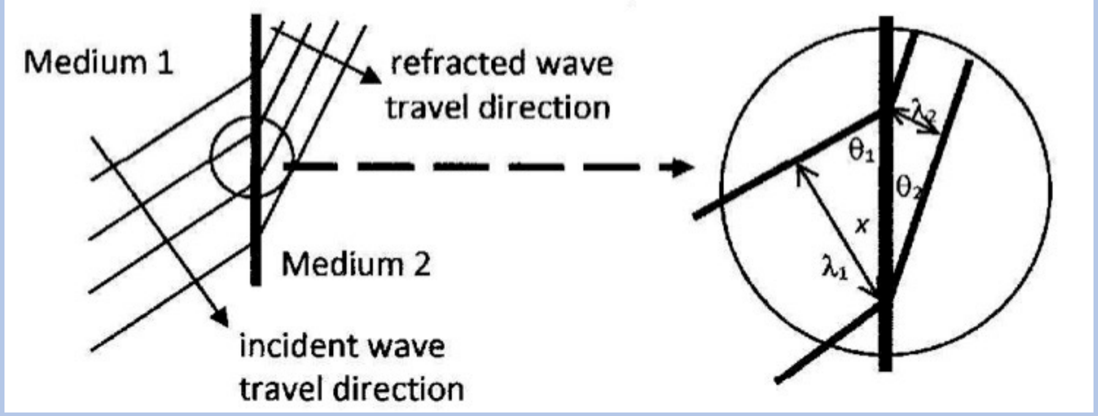
which is not observed.

Huygens’ Wave Model of Refraction (1)

In *Treatise on Light* (1690), Christiaan Huygens calculated light speed from Rømer’s observations.

He advanced the wave model for light travel in analogy with sound.

Waves refract when transitioning from one medium to another with a different wave speed, as in the diagram.



A light ray corresponds to the light travel direction.

Wavelength = λ , frequency = f

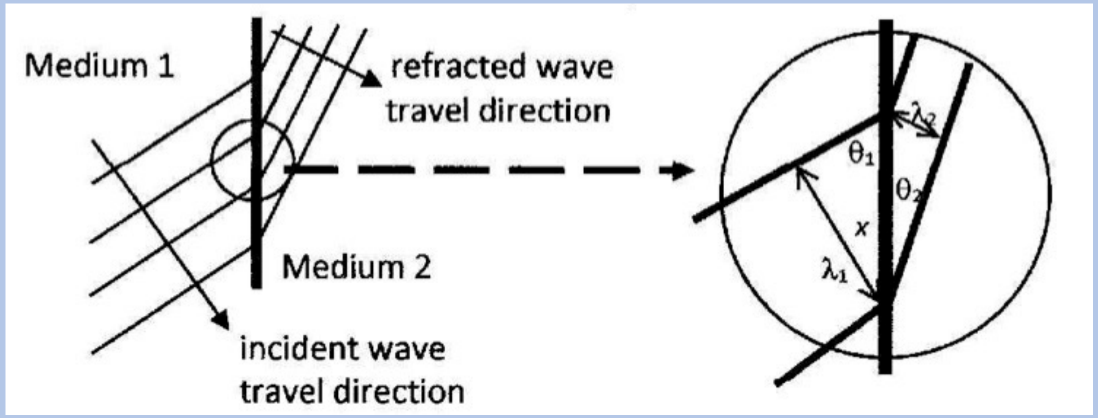
Wave speed = $v = \lambda f$

Huygens' Wave Model of Refraction (2)

Wave frequency *f* , *i. e.* number of waves per second at a point, is the same in both media.

The distance *x* between wave crests along the boundary is the hypotenuse of both triangles in the diagram.

The angle between wave crest and boundary equals the angle between wave direction (ray) and normal to boundary.



$\sin\theta_1 = \lambda_1/x$ $\sin\theta_2 = \lambda_2/x$

$$\frac{\sin\theta_1}{\sin\theta_2} = \frac{\lambda_1/x}{\lambda_2/x} = \frac{\lambda_1 f}{\lambda_2 f} = \frac{v_1}{v_2} = n_{12}$$

Fermat’s Principle of Least Time (1)

Pierre de Fermat (1607 – 1665), lawyer and councilor at the Parlement de Toulouse (1620 – 1665), was an able mathematician in his spare time.

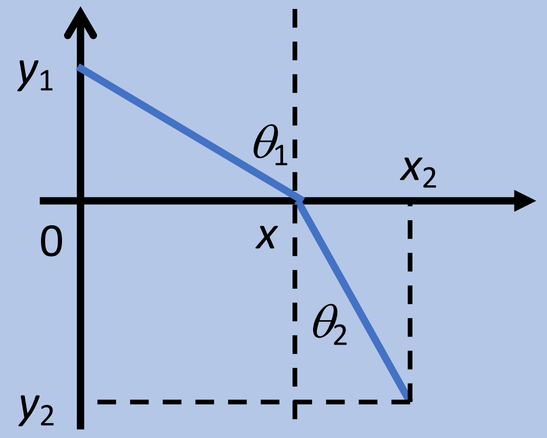
In 1657 Fermat received a letter from a friend noting that Heron’s principle of least distance did not explain light refraction.

In 1662 he replied that light refraction could be explained if
(1) light traveled slower in a higher index medium and
(2) the light path took the least time, not the least distance.

Fermat's Principle of Least Time (2)

Suppose light travels from point $(0, y_1)$ in Medium 1 (y>0) to a point on the boundary $(x, 0)$ to point (x_2, y_2) in Medium 2 (y<0).

We want to find the path with the least travel time.



Index of refraction from vacuum to Medium 1 = n_1.

Index of refraction from vacuum to Medium 2 = n_2

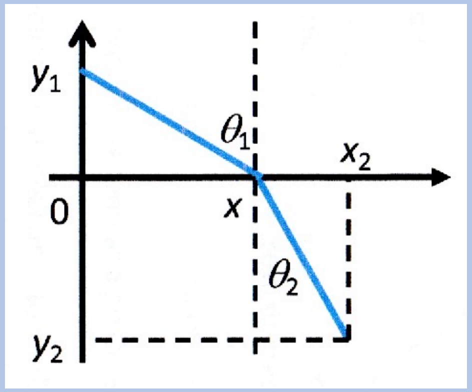
Light speed in Medium 1:
$v_1 = c/n_1$

Light speed in Medium 2:
$v_2 = c/n_2$

Fermat’s Principle of Least Time (3)

Let T = travel time for light from (0, y1) to (x,, 0) to (x2, y2).

$$T = \frac{\left[x^2+y_1^2\right]^{1/2}}{v_1} + \frac{\left[(x_2-x)^2+y_2^2\right]^{1/2}}{v_2}$$



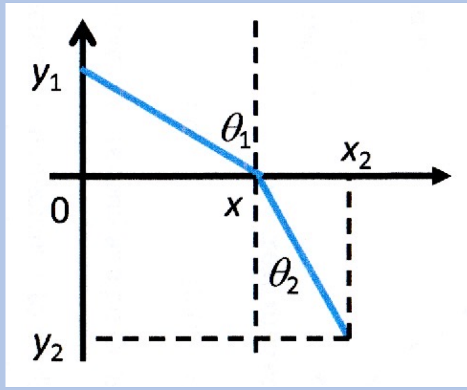
The *x*-value for the minimum time, is given by $\frac{dT}{dx} = 0.$

$$\frac{\frac{1}{2}[x^2 + y_1^2]^{-1/2}(2x)}{v_1} + \frac{\frac{1}{2}[(x_2 = x)^2 + y_2^2]^{-1/2}(-2)(x_2 = x)}{v_2} = 0$$

Fermat's Principle of Least Time (4)

Rearranging, we get

$$\frac{1}{v_1}\left[\frac{x}{[x^2+y_1^2]^{1/2}}\right]=\frac{1}{v_2}\left[\frac{(x_2-x)}{[(x_2-x)^2+y_2^2]^{1/2}}\right]$$



$$\frac{\sin\theta_1}{v_1}=\frac{\sin\theta_2}{v_2}$$

$$\frac{\sin\theta_1}{\sin\theta_2}=\frac{v_1}{v_2}=\frac{c/n_1}{c/n_2}=\frac{n_2}{n_1}=n_{12}=\text{constant}$$

Fermat’s Principle of Least Time (4)

Fermat’s Principle can, thus, explain straight-line travel, reflection, and refraction for light.

However, the principle was neglected at first because it opposed Descartes’ assumptions and later because it opposed Newton’s particle model for refraction.

The wave model for light was revived by Thomas Young and Augustin-Jean Fresnel in the early 19th century. Measurements of light speed in various media by Fizeau and Foucault in the mid-19th century confirmed the wave model predictions.