Plus ça change, plus c’est la même chose.* Conservation Laws and Special Relativity

*Jean-Baptiste Alphonse Karr, 1849

QuarkNet Workshop

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Constant Values in a Changing World

Gold and silver maintain nearly constant weight over time, even when melted and reshaped as coins. Constant weight implied constant value.

Conservation of Mass

Antoine Lavoisier (1743-1794), aided by his wife Marie-Anne, extended the constancy of weight to all chemical reactions, as measured by sensitive chemical balances.

Conservation of Motion

René Descartes (1596-1650) in *Principles of Philosophy* (1644) asserted that

“God . . . In the beginning created matter with both movement and rest, and now maintains in the sum total of matter, . . . the same quantity of motion and rest.”

This is the first statement of conservation principles (matter and motion) in physics.

Laws of Nature

Descartes elaborated on his conservation of motion principle with 3 “laws of nature” and 7 collision rules.

Laws of nature 1 and 2 were later combined in Newton’s first Law of Motion.

However, law 3 and the 7 collision rules that followed were soon noted to be at odds with the results of actual collisions.

Mechanical Philosophy

Despite those shortcomings, Descartes' "mechanical philosophy," that all change in motion should be explained by the "clear" and "distinct" actions of particle collisions, guided the thoughts of European natural philosophers until well after the triumph of Newton's *Principia* (1687).

17th Century Collision Rules (1)

European scientists agreed with Descartes mechanical philosophy but noted the failure of his collision rules. Consequently, the proper rules that governed collisions became a central problem.

17th Century Collision Rules (2)

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In answer to a 1668 call from the Royal Society of London to elucidate reliable rules for collisions, John Wallis, Christopher Wren, and Christiaan Huygens submitted responses.

17^{th} Century Collision Rules (3)

Huygens' submission was the most comprehensive.

In the absence of external forces,

(1) for perfectly elastic collisions, the total scalar quantity mv^2, later named "*vis viva*" by Gottfried Leibniz, is conserved, and

(2) for all collisions and explosions, the total vector quantity $m\mathbf{v}$, called the "quantity of motion" by Newton, is conserved.

18th Century Controversy (1)

18th century European and English scientists argued over whether mv^2 or mv was the "real" quantity of motion.

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Leibniz related the acquisition of mv^2 by a falling body to its height of fall. Willem ‘s Gravesande confirmed this experimentally by dropping brass balls into soft clay (1722).

Émilie du Châtelet (1706-1749)

Institutions de Physique (1740) reconciled ideas of Newton and Leibniz and emphasized mv^2 and its transformations to other forms of energy.

Her translation of Newton's *Principia* (1759) (with extensive commentary) is the standard French reference.

19^{th} Century Development (1)

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James Clerk Maxwell's kinetic theory of gases (1860s) related gas temperature to $mv^2/2$ of gas molecules.

William Thomson and Peter Tait established the terms "kinetic energy" for $mv^2/2$ and "momentum" for mv in *Treatise on Natural Philosophy* (1867).
They emphasized conservation of energy in physics.

20^{th} Century Problem

Walter Kaufmann, Alfred Bucherer, and others measured the kinetic energy and momentum of high energy electrons from radioactive β-decay in the early 1900s. They noted that $KE = mv^2/2$ and $\mathbf{p} = m\mathbf{v}$ did not apply unless they interpreted mass as a velocity dependent quantity.

The failure of $KE = mv^2/2$ is seen in an experiment by William Bertozzi with electrons in a linear accelerator at MIT (1962).

"The Ultimate Speed – An Experiment with High Energy Electrons"
https://www.youtube.com/watch?v=B0BOpiMQXQA

W. Bertozzi, *Am. J. Phys.*, **32**, 551-555 (1964).

20^{th} Century Question

Scalar kinetic energy ($mv^2/2$) and vector momentum ($m\mathbf{v}$) are important and independent quantities related to motion. Energy and momentum are each conserved under appropriate circumstances. Angular momentum, also a vector quantity, is a third conserved quantity of motion for particle interactions.

Why is nature constrained by <u>three</u> conserved quantities of motion?

Emmy Noether (1882- 1935)

Emmy Noether answered the question in her 1915 paper.

Einstein wrote in the *New York Times* on May 4, 1935, in memorial, "In the judgment of the most competent living mathematicians, Fraeulein Noether was the most significant creative mathematical genius thus far produced since higher education of women began."

Yet, few are aware of her work.

Plus ça change, plus c'est la même chose. (1)

The more things change, the more they stay the same.

Symmetry:

In life – "harmonious balance and beautiful proportions"

In physics – "change without change"

Examples:

<u>Continuous symmetry</u>: rotation of a circle about its center by any angle

<u>Discrete symmetry</u>: rotation of an equilateral triangle about its center by (n x 120) degrees for n = integer

Plus ça change, plus c'est la même chose. (2)
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In 1915 Emmy Noether proved that every continuous symmetry in the laws of physics implies a conserved quantity:

(1) time translation symmetry implies energy conservation,

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In 1915 Emmy Noether proved that every continuous symmetry in the laws of physics, there is a conserved quantity:

(1) time translation symmetry implies energy conservation,

(2) space translation symmetry implies momentum conservation,

(3) space rotation symmetry implies angular momentum conservation.

Conserved and Invariant Quantities

Conserved quantities in physics are those which remain the same from before to after some physical interaction, *e. g.* electric charge, energy, momentum.

Invariant quantities in physics are those which remain the same when measured in different inertial reference frames moving at constant velocity (boosted) relative to each other, *e. g.* electric charge, mass.

Conserved Quantities and Invariant Laws

Noether's theorem explains why energy, momentum, and angular momentum are locally conserved in physical interactions. But the theorem does not specify the formulation of the conserved quantities.

We want invariant laws of physics, *i. e.* the same in all inertial reference frames. And we want energy and momentum, though not invariant quantities, conserved in all inertial reference fames.

Newton's Laws and Galilean Transformations

Rocket system (t', x', y', z') origin and axes overlap Lab system (t, x, y, z) origin and axes at time $t = t' = 0$ and moves with constant velocity v_R in the +x-direction.

Galilean Transformations: $t = t'$, $x = x' + v_R t'$, $y = y'$, $z = z'$

Velocities: $\frac{dt}{dt'} = 1,\quad v_x = \frac{dx}{dt} = \frac{d(x' + v_R t')}{dt'} = \frac{dx'}{dt'} + v_R$

$$v_x = v_{x'} + v_R\ ,\quad v_y = v_{y'}\ ,\quad v_z = v_{z'}$$

Accelerations: $a_x = a_{x'}\ ,\quad a_y = a_{y'}\ ,\quad a_z = a_{z'}$

Forces: For $m = m'$, $F_x = F_{x'}\ ,\quad F_y = F_{y'}\ ,\quad F_z = F_{z'}$ INVARIANT

Newton's Laws and Lorentz Transformations

Consider the usual Rocket system (t', x', y', z') and Lab system (t, x, y, z).
$\gamma = (1-v_R^2)^{-1/2}$, c = 1 units, v_R = Rocket system velocity along +x-axis.

Lorentz Transformations: $t = \gamma(t' + v_R x')$, $x = \gamma(x' + v_R t')$, $y = y'$, $z = z'$

Velocities: $\frac{dt}{dt'} = \gamma(1 + v_R v_{x'})$, $\frac{dx}{dt'} = \gamma(v_{x'} + v_R)$

$$v_x = \frac{dx/dt'}{dt/dt'} = \frac{\gamma(v_{x'}+v_R)}{\gamma(1+v_R v_{x'})} = \frac{v_{x'}+v_R}{1+v_R v_{x'}}, \quad v_y = \frac{v_{y'}}{\gamma(1+v_R v_{x'})}, \quad v_z = \frac{v_{z'}}{\gamma(1+v_R v_{x'})}$$

Accelerations: $a_x \neq a_{x'}$, $a_y \neq a_{y'}$, $a_z \neq a_{z'}$

Forces: For $m = m'$, $F_x \neq F_{x'}$, $F_y \neq F_{y'}$, $F_z \neq F_{z'}$ NOT INVARIANT

Brief Aside on Velocity Composition (1)

$$v_x = \frac{v_{x'} + v_R}{1 + v_{x'} v_R}$$

Earth orbital speed = 30 km/s = 10^{-4} *c*

For $v_R = v_{x'} = 10^{-4}$, v_x = ?

For v_R = 0.5 and $v_{x'}$ = 0.5, v_x = ?

For v_R = 0.5 and $v_{x'}$ = 1.0, v_x = ?

Brief Aside on Velocity Composition (2)

$$v_x = \frac{v_{x'} + v_R}{1 + v_{x'} v_R}$$

Earth orbital speed = 30 km/s = 10^{-4} *c*

For $v_R = v_{x'} = 10^{-4}$, $v_x = (2\text{x}10^{-4})(1 - 2\text{x}10^{-8})$, only 2 parts in 10^8 less than the Newtonian value.

[NOTE: Earth escape speed is 11.2 km/s]

For $v_R = 0.5$ and $v_{x'} = 0.5$, $v_x = \frac{0.5+0.5}{1+(0.5)(0.5)} = \frac{1.0}{1.25} = 0.8$

For $v_R = 0.5$ and $v_{x'} = 1.0$, $v_x = \frac{1.0+0.5}{1+(1.0)(0.5)} = \frac{1.5}{1.5} = 1.0$

Brief Aside on Velocity Composition (3)

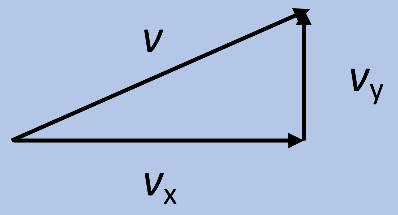
Show that a light flash moving along the +y'-axis of the Rocket frame with speed *c*=1, is observed to move along a diagonal line in the Lab frame with speed *c*=1. [NOTE: For this calculation, $v_{x'} = 0$, $v_{y'} = 1$.]

Recall: $v_x = \frac{v_{x'}+v_R}{1+v_R v_{x'}}$, $v_y = \frac{v_{y'}}{\gamma(1+v_R v_{x'})}$, $\gamma = \frac{1}{\sqrt{1-v_R^2}}$

Brief Aside on Velocity Composition (4)

Show that a light flash moving along the +y'-axis of the Rocket frame with speed *c*=1, is observed to move along a diagonal line in the Lab frame with speed *c*=1. [NOTE: For this calculation, $v_{x'} = 0$, $v_{y'} = 1$.]

Recall: $v_x = \frac{v_{x'} + v_R}{1 + v_R v_{x'}}$, $v_y = \frac{v_{y'}}{\gamma(1 + v_R v_{x'})}$, $\gamma = \frac{1}{\sqrt{1 - v_R^2}}$



$$v^2 = v_x^2 + v_y^2 = \left[\frac{0 + v_R}{1 + 0}\right]^2 + \left[\frac{1}{\gamma(1 + 0)}\right]^2 = v_R^2 + (1 - v_R^2)$$

$v^2 = 1$, $v = 1$ **Speed of light in vacuum is invariant.**

Dilemma (1)

Newton's Laws of Motion for mechanics, based on his definition of momentum (*mv*), are invariant under Galilean transformations but not under Lorentz transformations.

Dilemma (2)

Newton's Laws of Motion for mechanics, based on his definition of momentum ($m\mathbf{v}$), are invariant under Galilean transformations but not under Lorentz transformations.

Lorentz, Poincaré, and Einstein showed that Maxwell's equations for electromagnetism are invariant under Lorentz transformations but not under Galilean transformations.

Dilemma (3)

Newton’s Laws of Motion for mechanics, based on his definition of momentum ($m\mathbf{v}$), are invariant under Galilean transformations but not under Lorentz transformations.

Lorentz, Poincaré, and Einstein showed that Maxwell’s equations for electromagnetism are invariant under Lorentz transformations but not under Galilean transformations

Many experiments, *e. g.* Michelson-Morley, muon time dilation, electron speed measurements, show that nature conforms to Lorentz transformations. How must we change our laws of mechanics?

Redefinition (1)

We need to redefine energy and momentum for particles to

(1) maintain conservations principles,

(2) remain compatible to Lorentz transformations, and

(3) approximate the Newtonian definitions at speeds small compared to light speed.

Redefinition (2)

20th Century

Keep the Newtonian definition of momentum (*mv*) by treating mass as velocity dependent. This led to the expressions "longitudinal mass", "transverse mass", "relativistic mass".

Redefinition (3)

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Keep the Newtonian definition of momentum (*m***v**) by treating mass as velocity dependent. This led to the expressions “longitudinal mass”, “transverse mass”, “relativistic mass”.

21st Century

Treat mass an an intrinsic, invariant property of an object and change the definition of of momentum and kinetic energy.

Redefinition (4)

Einstein (1905) proposed $KE = m(\gamma\text{-}1)$ for a particle in motion and $E = m$ for a particle at rest, *i. e.*
a particle's rest energy equals its inertia. (c = 1 units)

Total energy of a moving particle is

$E = m + KE = m + m(\gamma\text{-}1) = \gamma m$.

Does this account for Bertozzi's measurements?

Redefinition (5)

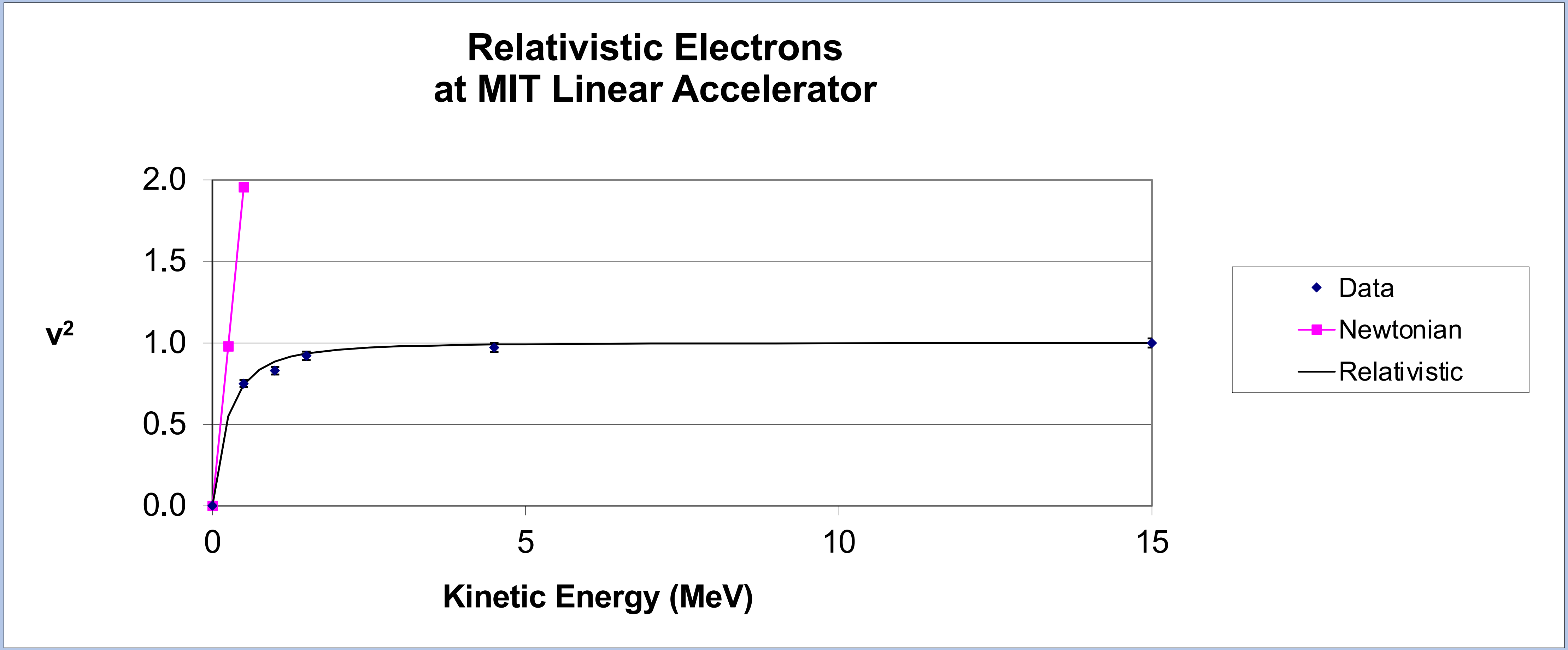
On *The Ultimate Speed Student Spreadsheet*, work through one row of calculations for #1.

Then work through items #2 and #4.

Bertozzi Data

MEASUREMENTS			CALCULATIONS							THEORY (dimensionless)	
KE	Δt	[+/-]	Δt	[+/-]	$v = \Delta d/\Delta t$	[+/-]	v	v^2	[+/-]	Newtonian	Relativistic
(Mev)	(division)	(division)	(10^{-8} s)	(10^{-8} s)	(10^8 m/s)	(10^8 m/s)	(m/m)			$v^2 = 2KE/m_e$	$v^2 = 1 - (m_e/(m_e+KE))^2$
0	0		0		0		0	0		0	0
0.5	3.30	0.04	3.23	0.0392	2.60	0.03	0.866	0.751	0.019	1.96	0.745
1.0	3.14	0.04	3.08	0.0392	2.73	0.04	0.911	0.829	0.022	3.91	0.886
1.5	2.98	0.04	2.92	0.0392	2.88	0.04	0.959	0.921	0.025	5.87	0.935
4.5	2.90	0.04	2.84	0.0392	2.96	0.04	0.986	0.972	0.027	17.61	0.990
15.0	2.86	0.04	2.80	0.0392	3.00	0.04	1.000	0.999	0.029	58.71	0.999

Bertozzi Plot



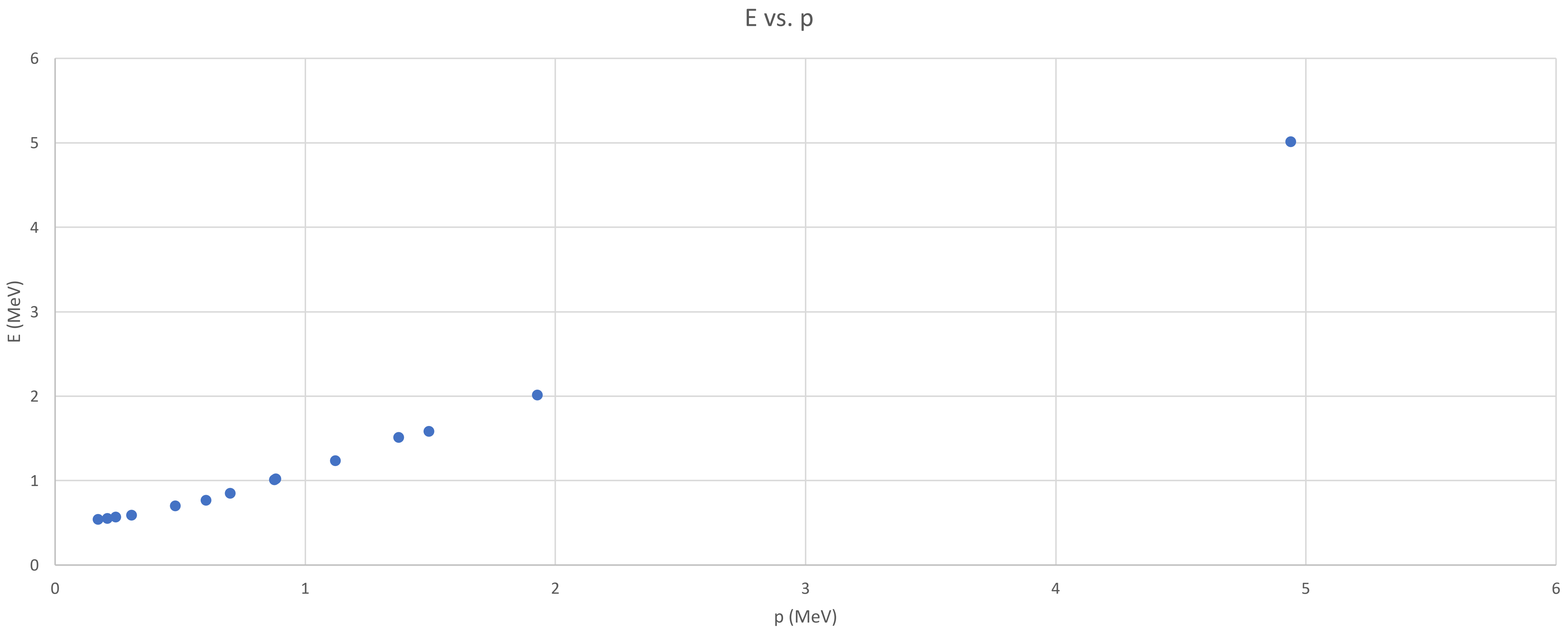
Kaufmann, Bucherer, and Bertozzi

When Walter Kaufmann and Alfred Bucherer measured energy and momentum of high-speed electrons from radioactive β-decay in the early 1900s, they found that simple Newtonian expressions did not match the data.

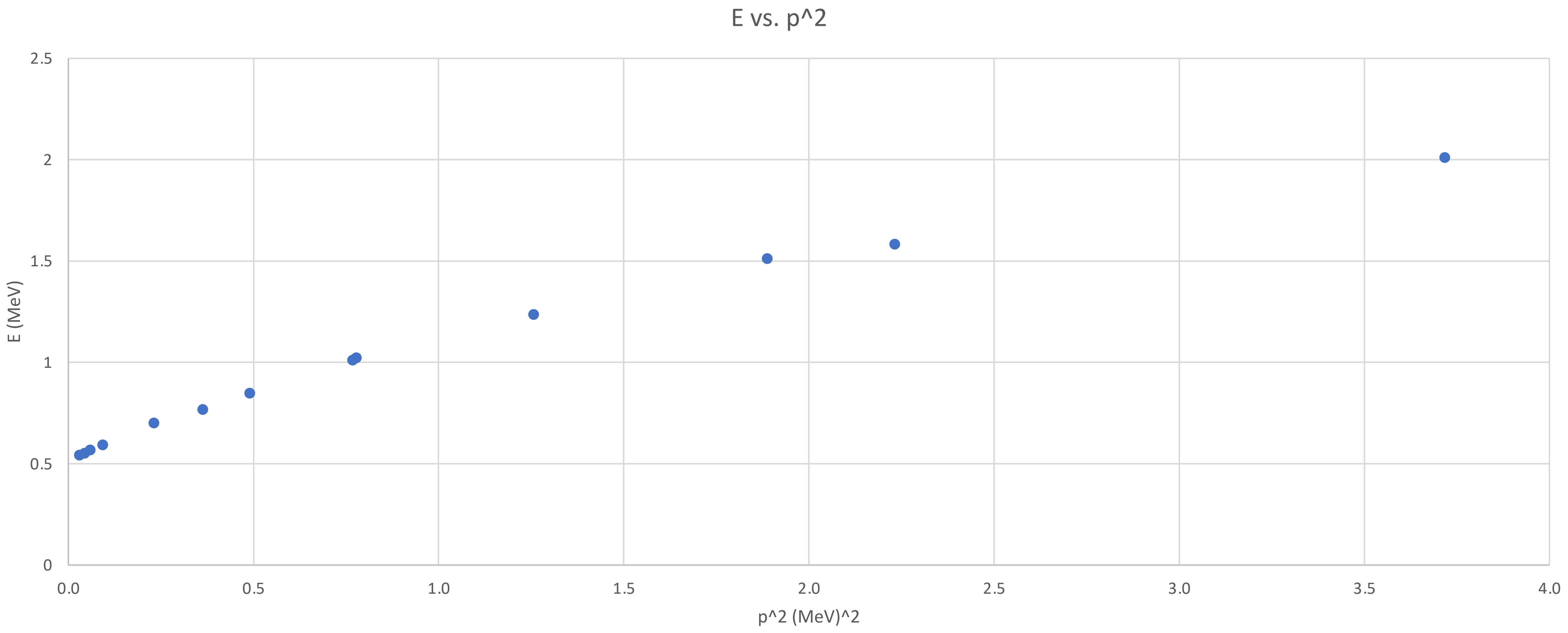
The QuarkNet *Energy, Momentum, Mass* Data Activity asks students to plot some of Kaufmann and Bucherer’s data to find a relation between E, p, and m.

Plots incorporating some of Kaufmann’s, Bucherer’s, and Bertozzi’s data up to KE = 1.5 MeV follow.

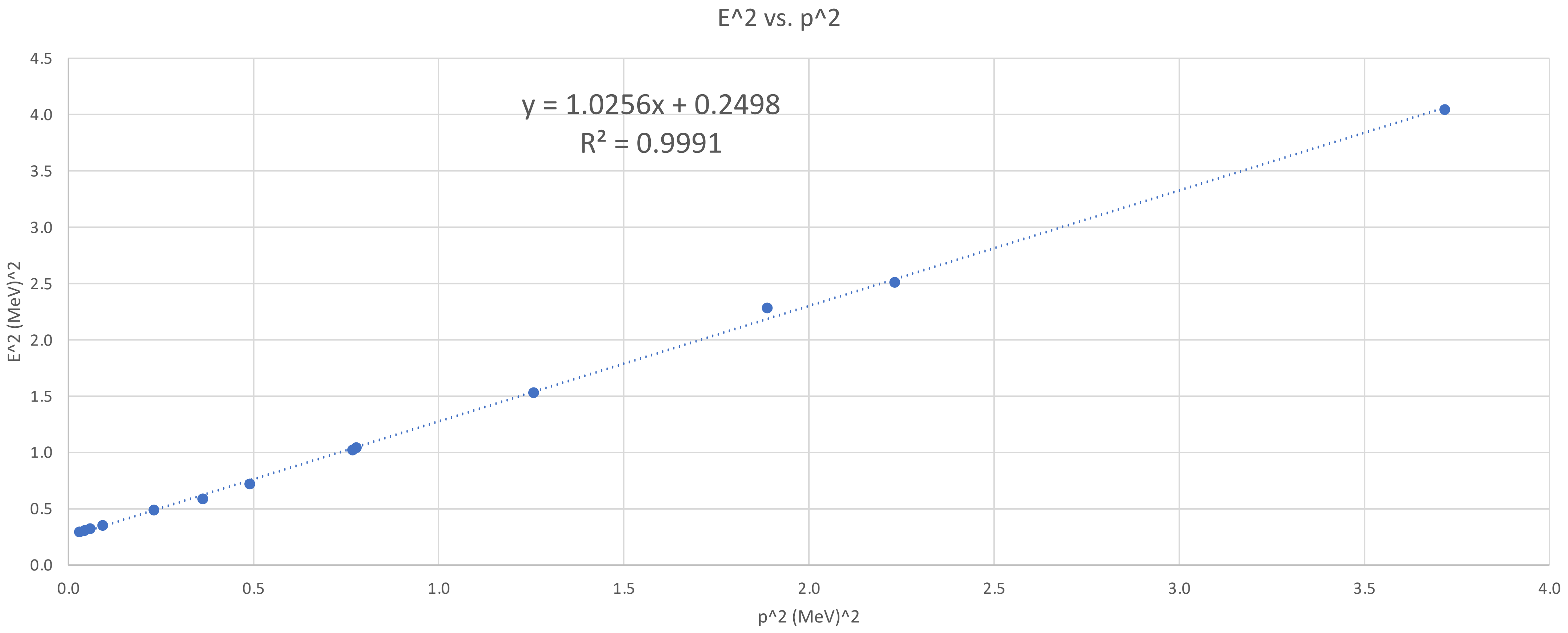
Kaufmann, Bucherer, Bertozzi Data (1)



Kaufmann, Bucherer, Bertozzi Data (2)



Kaufmann, Bucherer, Bertozzi Data (3)



Redefinition (6)

Electron mass is 0.511 MeV/c^2.

The straight line on the final plot is well represented by
$E^2 = m^2 + p^2$.

Exercise: Combine $E = \gamma m$ and $E^2 = m^2 + p^2$ to show $p = \gamma m v$ for particles with mass.

For photons ($m = 0$), $E = p$ (c=1 units)
or $E = pc$ (conventional units).

Redefinition (7)

To provide more theoretical underpinning to the $m^2 = E^2 - p^2$ relationship, start with the **invariant spacetime interval** between two events $\Delta\tau^2 = \Delta t^2 - (\Delta x^2 + \Delta y^2 + \Delta z^2)$. The proper time, $\Delta\tau$, is the time measured by a clock in an inertial frame where the two events take place at the clock position.

Suppose the two events mark the infinitesimal change in position of a particle of mass *m* moving with velocity ***v*** in the +*x*-direction of the Lab system. Then $dy = 0$, $dz = 0$, and $dx = vdt$.

$$d\tau^2 = dt^2 - dx^2 = dt^2 - (vdt)^2 = (1-v^2)\mathrm{d}t^2 = (1/\gamma)^2 dt^2 \quad (c=1 \text{ units}).$$

Redefinition (8)

Then $\gamma^2 = \left(\frac{dt}{d\tau}\right)^2$.

Dividing $d\tau^2 = dt^2 - dx^2$ by $d\tau^2$, we get

$$1 = \left(\frac{dt}{d\tau}\right)^2 - \left(\frac{dx}{d\tau}\right)^2 = \left(\frac{dt}{d\tau}\right)^2 - \left(\frac{dt}{d\tau}\frac{dx}{dt}\right)^2 = \gamma^2 - \gamma^2 v^2 .$$

Multiplying by m^2 yields, $m^2 = \gamma^2 m^2 - \gamma^2 m^2 v^2 = E^2 - p^2$.

Redefinition (9)

With our new definitions of particle energy ($E = \gamma m$) and momentum ($\boldsymbol{p} = \gamma m\mathbf{v}$), E and $\boldsymbol{p}$ transform between reference frames with Lorentz transformations analogous to the spacetime transformations, where E replaces t and p_x replaces x.

The particle mass m is an invariant,
the same in all reference frames.

Low Velocity Limit (1)

The definitions of particle energy ($E = \gamma m$) and momentum ($\boldsymbol{p} = \gamma m \boldsymbol{v}$), work well at high velocities, as measured by Kaufmann, Bucherer, Bertozzi, and many others.

How do those definitions apply at low velocities?

Low Velocity Limit (2)

Binomial expansion for x<1: $(1-x)^{-1/2} = 1 + \frac{1}{2}x + \frac{3}{8}x^2 \cdots$

c = 1 units:

$$E = \gamma m = (1 - v^2)^{-1/2} m \cong \left(1 + \frac{1}{2}v^2\right) m = m + \frac{1}{2}mv^2$$

Conventional units:

$$E \cong mc^2 + \frac{1}{2}(mc^2)\left(\frac{v}{c}\right)^2 = mc^2 + \frac{1}{2}mv^2$$

Low Velocity Limit (3)

Binomial expansion for x<1: $(1-x)^{-1/2} = 1 + \frac{1}{2}x + \frac{3}{8}x^2 \cdots$

c = 1 units:

$$p = \gamma m v = (1-v^2)^{-1/2} m v \cong \left(1 + \frac{1}{2}v^2\right) m v \cong m v$$

Conventional units:

$$pc \cong (mc^2)\left(\frac{v}{c}\right) \text{ or } \quad p = mv$$

At low velocities, we have traditional Newtonian expressions!

Redefinition (10)

Our new definitions of particle energy ($E = \gamma m$) and momentum ($\boldsymbol{p} = \gamma m \boldsymbol{v}$) fulfill all our requirements. They

(1) maintain conservations principles,

(2) remain compatible to Lorentz transformations, and

(3) approximate the Newtonian definitions at speeds small compared to light speed.