Worlds That Aren't

$F_G = -\frac{G_N m_1 m_2}{r^2}$ ← why squared? why 2? why not 1.9/2.1 etc?

$m\ddot{\vec{r}} = F(x, t, \dot{\vec{r}}$ etc)

Newton: clockwork; Deterministic

Recall: point mass / point charge:

field lines get less dense as $\frac{1}{r^2}$ in 3D space

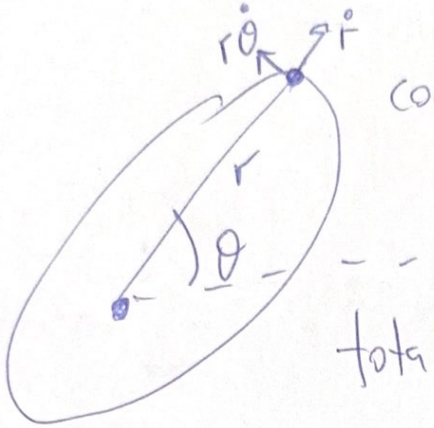
in 2D: field is $\frac{1}{r}$ (like gauss for cylindrical)

in 1D: constant

$\therefore$ if d is dimension, $F \propto \frac{1}{r^{d-1}}$ (in 4D, $\frac{1}{r^3}$)

Interesting: only w/ $\frac{1}{r^2}$ can you have stable orbits; i.e. that return to original spot

Conservation laws: E, $\vec{L}$ conserved



components of velocity in polar coordinates: $\dot r$, $\dot\theta r$

$$\text{total } E = \frac{1}{2}m\,\text{"}v\text{"}^2 - U(r)$$

$$= \frac{1}{2}m\left(\dot r^2 + r^2\dot\theta^2\right) - \frac{c_1}{r^p}$$

$$L = mr\,r\dot\theta = mr^2\dot\theta$$

$\dot\theta = \frac{L}{mr^2}$ i.e. angular speed goes up w/ decreasing r

$$E = \frac{1}{2}m\left(\frac{L}{mr^2}\right)^2 r^2 - \frac{\text{const}}{r^p} \quad (p \text{ is exponent})$$

$$= \frac{1}{2}m\dot r^2 + \left(\frac{1}{2}\frac{L^2}{mr^2} - \frac{c}{r^p}\right)$$

now, the $\dot\theta$ kinetic looks like a potential term!

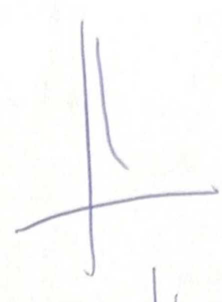
$\Downarrow$

if only $\frac{L}{mr^2}$,

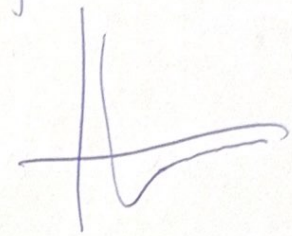
: straight line motion

(AKA p=0)

if $p=1$: $\frac{-c}{r^1}$ wins for small r

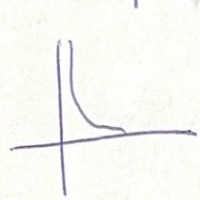


and $\frac{L^2}{mr^2}$ wins for big r

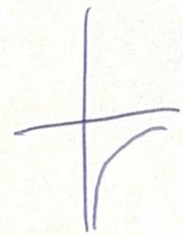


if $p=2$: both terms are $\frac{1}{r^2}$

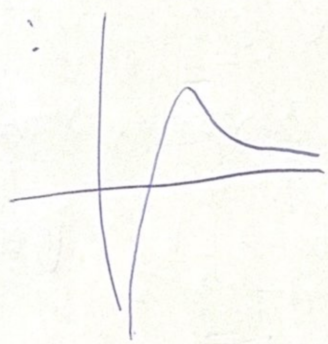
$\Rightarrow$



or



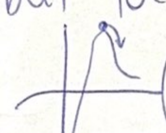
if $p=2.1$:



(why?...)

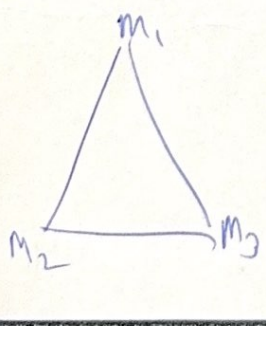
in $p=1$, circular orbit is stable: local minimum

in $p=2.1$, you got a circle, but local max! unstable!



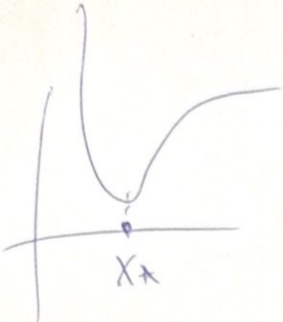
(give it a little $\dot{r}$)

Somewhat tangential:



triangle const ω?

yes if equilateral triangle! even if $m_1 \neq m_2 \neq m_3$



remember: local min: leading nonzero term is $\frac{1}{2}kx^2$ (parabola)

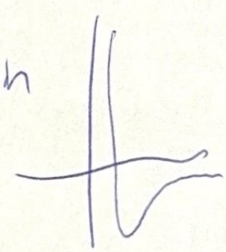
displace from X_* to δx, then do Taylor expansion

for $U(x)$: $\underset{\text{AKA } x_* + \delta x}{\underset{\downarrow}{U(x_*)}} \approx \underbrace{U(x_*)\cancel{x}^0}_{0!} + U'\underbrace{(x_*)\overset{(x_*+\delta x)}{\cancel{x}}}_{1!} + \frac{1}{2}U''(x_*)\cancel{x}^2_{(x_*+\delta x)^2}$

first two terms are $\underline{\text{zero}}$: first because it's an arbitrary constant; second b/c local min

but remember this is all for $p=1$ $\quad U_{eff} = \frac{1}{2}\frac{L^2}{mr^2} - \frac{c}{r}$

$U'(r) = \frac{-L^2}{mr^3} + \frac{pc}{r^{p+1}} = 0$ at local min



for r_{circ}: $r_{circ}^{p-2} = \frac{mpc}{L^2}$

remember $mr_{circ}^2\dot{\theta} = L$ so $\dot{\theta} = \frac{L}{mr_{circ}^2} = \frac{\sqrt{\frac{mpc}{r_{circ}^{p-2}}}}{mr_{circ}^2} = \frac{\text{const.}}{r^{\left(2+\frac{p-2}{2}\right)}}$

$\cancel{r_{circ}^2 = \frac{L}{m\dot{\theta}}}$

$r_{circ} = \left(\frac{mpc}{L^2}\right)^{p-2}$ or $L = \sqrt{\frac{mpc}{r_{circ}^{p-2}}}$

so $\omega = \sqrt{\frac{k}{m}}$ for $p=1$, $\omega = \text{const}\frac{1}{r^{3/2}}$ kepler!

$r_{circ}^{p-2} = \frac{mpc}{L^2}$, $mr^2\dot{\theta} = L \Rightarrow \dot{\theta} = \frac{L}{mr_{circ}^2}$; $\dot{\theta}^2 = \frac{L^2}{m^2 r_{circ}^p}$

$= \omega^2$

for small displacemnt, $\omega_{s.o.}^2 = \text{const} \cdot \frac{U''(r_{circ})}{m}$

$$U'_{eff}(r) = \frac{-\cancel{L}^2}{mr^3} + \frac{pc}{r^{p+1}} = 0$$

$$V''_{eff}(r) = \frac{3L^2}{mr^4} - \frac{p(p+1)c}{r^{p+2}} \text{ so } \omega_{s.o.}^2 = \frac{3L^2}{m^2 r^4} - \frac{p(p+1)c}{mr^{p+2}}$$

$= \text{eww}.$

if $p=1$: $\omega_{circ}^2 = \frac{L^2}{m^2 r_{circ}^4} = \frac{L^2}{m^2}$ and $r_{circ}^{-1} = \frac{mpc}{L^2}$

$$\text{so } \omega_{circ}^2 = \left(\frac{mc}{L}\right)^4 = \frac{m^2 c^4}{L^6}$$

$\omega_{s.o.}^2 = \frac{3L^2}{m^2 r^4} - \frac{p(p+1)c}{mr^{p+2}}$ when $p=1$, $\omega_{s.o.}^2 = \omega_{circ}^2$!

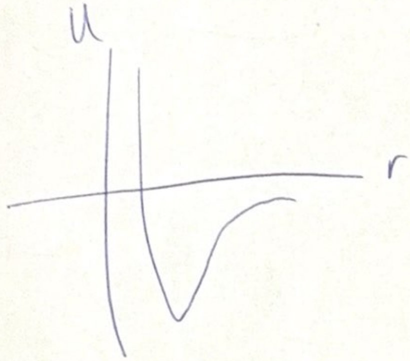
for $p \neq 1$, $\omega_{s.o.}^2 = f(p)\,\omega_{circ}^2$.

Over to matter & atom stability

total $E = \frac{p^2}{2m} - \frac{ke^2}{r}$

Uncertainty: $\Delta p \Delta r \sim \hbar \quad p \approx \Delta p \sim \frac{\hbar}{r}$

so $E \sim \frac{\hbar^2}{2mr^2} - \frac{ke^2}{r}$ so classical L^2 is now $\hbar^2$ which recalls Bohr: $L = n\hbar$



r_{atom}: $\frac{\hbar^2}{mr^2} \sim \frac{ke^2}{r} \Rightarrow r_{atom} \sim \frac{\hbar}{kme^2}$

so $p=1$ is built into atom stability! (won't work in >3D either!)

$E = \frac{|\vec{p}|^2}{2m} - \frac{ke^2}{r^{1+p}}$ now, p is deviation from 1

P.S. used to could only test gravity to 1cm because of Casimir / Van de Waals shit.

Now w/ some clever tricks, ~50 μm with screening, rotating plates etc.

Counterfactual 2 : $F \neq ma$

$$m\ddot{x} = F_{Newton} \Rightarrow m\dot{x} = f_{Nima}?$$

no Noether-type time symmetry here.
also no inertial frames.

how about $m\dddot{x} = f_{Nima}'$? No bueno either.

$$m\ddddot{x} = f_{Nima}''?$$

if you want any hope of stability here,
you need block-spring-type $F \sim -kx$
(4th derivative: $-$ sign goes away if sin/cos!)

in our world, $\omega^2 = \frac{k}{m}$ and $x(t) = A\cos\omega t$

in new world, $\ddddot{x} = \omega^4 A\cos\omega t$

ω is imaginary! $x \sim \cosh/\sinh$!
bye-bye particle!

even if you fix negative ω $x^{(6)}$, only two
solutions of 6 are oscillatory; other 4 are
complex

Counterfactual 3

Why is energy conserved? $m\ddot{x} = F(x)$

1-D: don't mess w/ oscillator; energy dumped in

Noether: Energy cons. $\Leftrightarrow$ time symmetry

$$F = \frac{-du}{dx}, \quad m\ddot{x} = \frac{-du}{dx}$$

dirty trick: $\dot{x}\left[m\ddot{x} + \frac{du}{dx}\right] = 0$

$$\frac{d}{dt}\left(\frac{1}{2}m\dot{x}^2 + u(x)\right) = 0$$

2-D: $m\ddot{x} = F_x$; $m\ddot{y} = F_y$; $F_x = \frac{-\partial}{\partial x}u$

$$F_y = \frac{-\partial}{\partial y}u$$

You can't guarantee that F is the derivative of something, the way you can in 1-D

$$\frac{\partial}{\partial y}F_x = \frac{\partial}{\partial y}\frac{\partial}{\partial x}F \;;\; \frac{\partial}{\partial x}F_y = \frac{\partial}{\partial x}\frac{\partial}{\partial y}F_y$$

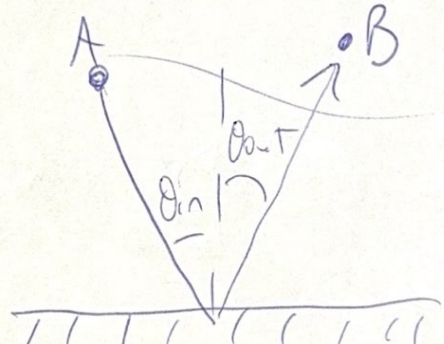
equal?? maybe not!

$$m\ddot{x} + \gamma\dot{x} + kx = 0$$

↑ friction! not time-symmetric!

matter: model a rock as a bunch of blocks & springs and it's hard to prove you can treat the rock with the same $F=ma$!

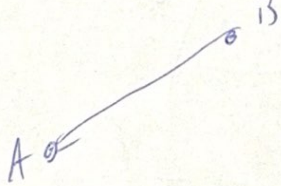
if the "jiggles" are comparable to potential energy bumps, for example, no bueno. (quantum gear?)



try other ways to bounce from A to B, all paths are longer.

Principle of Least Action

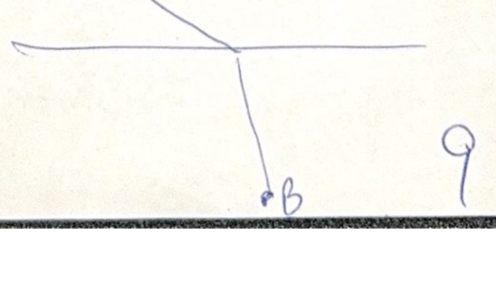
Light:



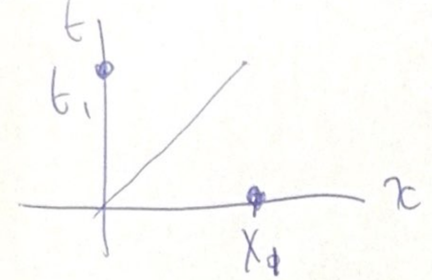
shortest time

Lifeguard problem!

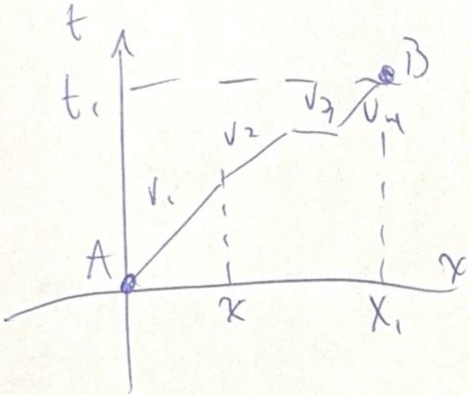
$$\frac{1}{V_{sand}} \sin\theta_{sand} = \frac{1}{V_{water}} \sin\theta_{water}$$



No forces, 1-D:



straight line, const v.



What are we minimizing here? Average kinetic energy!

Shortest path: $K = \frac{1}{2}m\left(\frac{X_1}{t_1}\right)^2$

"average" of triangle is $\frac{1}{2}m\left(\frac{X_1}{t_1}\right)^2 t_1$

$= \frac{\frac{1}{2}m X_1^2}{t_1}$

2-part kinked path: $\frac{1}{2}m\,\frac{T}{2}\left(\frac{X}{T/2}\right)^2 +$

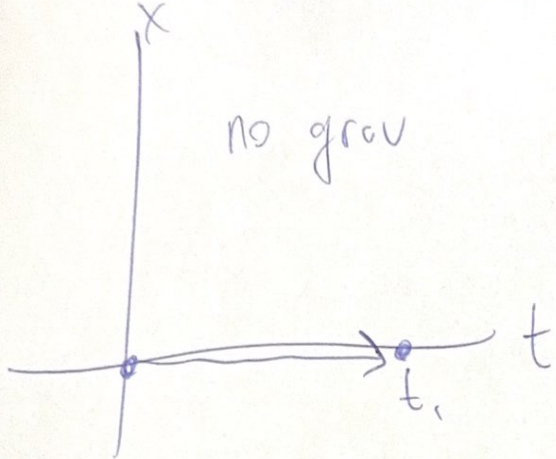
$+ \frac{1}{2}m\,\frac{T}{2}\left(\frac{X_1 - X}{T/2}\right)^2$

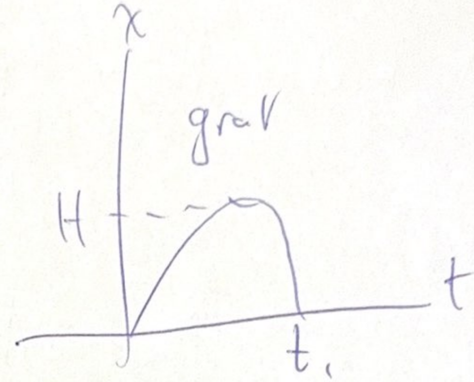
$= \frac{1}{2}m\,\frac{1}{2T}\left[4x^2 + 4(X-X_1)^2\right] = \frac{1}{2}\frac{m}{T}\left[2x^2 + 2(X-X_1)^2\right]$

obviously x has to be smaller than X_1, so we can find a min x and minimum K

$4x - 4(X - X_1) = 0$ so $x = X_1/2$

particle "sniffs out" all possible paths & chooses shortest path (??!!)





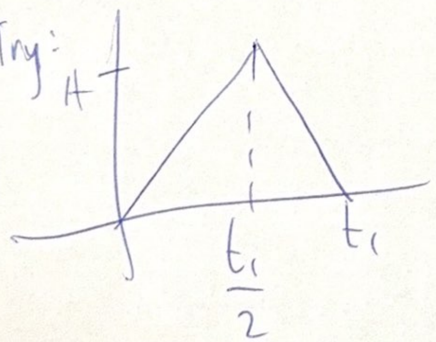
$H = \frac{1}{2} g \left(\frac{t_1}{2}\right)^2$ at time $t_2 = \frac{1}{2} t_1$

at $t = \frac{1}{4} t_1$, $H = \frac{1}{2} g \left(\frac{\frac{t_1}{2}}{2}\right)^2 = \frac{1}{2} g \left(\frac{t_1^2}{4}\right)$

$= \frac{3}{32} g t_1^2$ (?)

Guess that we <u>minimize</u> Average KE - Average PE

Try:



Av $K = \frac{1}{2} m \left(\frac{H}{T/2}\right)^2 \frac{T}{2} \cdot 2$

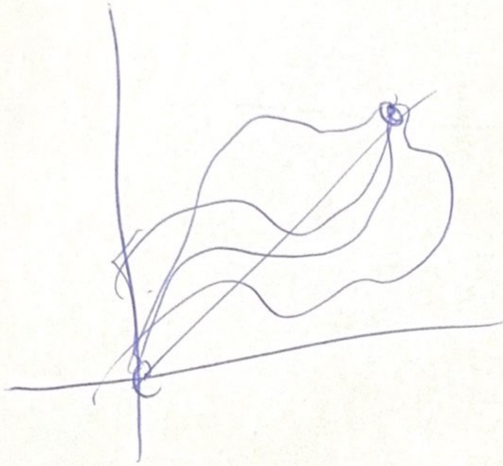
slope·dist   down

$= \frac{2mH^2}{t_1}$

Av $U = \frac{mgHT}{2}$ "Action" = K-U

$K-U = \frac{2mH^2}{T} - \frac{mgHT}{2}$ minimize K-U=0 min K-U : $H = \frac{1}{8} g t_1^2$

~~$\int$~~ K-U : "action" (S)



Amp A→B

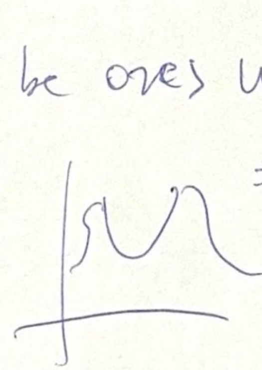
$$\sum_{\text{paths}} e^{iS/\hbar} \qquad S = \int dt\,(K-U)$$

S = time · Energy

$\hbar$: time · Energy

oscillating phases:
most paths cancel

only ones that don't will be ones where the ~~energy~~ action doesn't change much



= at a local min!

so minimum action overall you can get from local minimum action

$$S[x(t)] = \int_0^T dt\left(\frac{1}{2}m\dot{x}^2 - U(x)\right)$$

at min, nearby paths are same action

"functional"

$$S[x_1(t) + \delta x(t)] \approx S[x_1(t)] + \mathcal{O}(\delta x^2)$$

$$S = \int dt\, \tfrac{1}{2}m(\dot{x}_{cl} + \delta\dot{x})^2 - U(x_{cl} + \delta x)$$ x_{cl} = classical

$$= S[x_{cl}] + \int dt\, [m\dot{x}_{cl}\,\delta\dot{x} - \delta x\, U'(x_{cl})]$$ (??)

$U' = ma$

integrate by parts

$$= \int dt \left[\frac{d}{dt}(m\dot{x}_{cl}\delta x - m\ddot{x}_{cl}\delta x)\right]$$

$\delta_x, \delta_x = 0$

near $x = 0$,
$x = x_1$?

$$\int dt\, m\dot{x}_{cl}\,\delta\dot{x} = \int dt[(m\dot{x}_{cl}\delta x)' - m\ddot{x}_{cl}\delta x]$$

$$= \int dt \frac{d}{dt} m\dot{x}_{cl}\delta x = \underbrace{m\dot{x}_{cl}\delta x(t = t_1)}_{=0\ (??)} - \underbrace{m_{cl}\delta x(t=0)}_{=0\ (??)}$$

$$S = \int dt - \delta x[\cancel{\delta x}\, m\ddot{x}_{cl} + U'(x_{cl})]$$ is minimized when $S = 0$

So $m\ddot{x}_{cl} + U'(x_{cl}) = 0$

$F = ma$!